

PH107 Tutorial Solutions

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Tutorial 1

Useful formulae for this tut: Planck's law

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp(\frac{hc}{\lambda k_B T}) - 1} d\lambda \quad (1)$$

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(\frac{h\nu}{k_B T}) - 1} d\nu \quad (2)$$

Note that $u(\nu)$ is energy density per unit frequency and $u(\lambda)$ is per unit wavelength and so the quantities $u(\nu)d\nu$ and $u(\lambda)d\lambda$ represent the same thing which is increment of energy density for the small change in that variable however these variables change in opposite signs so for showing equality we have $u(\lambda)d\lambda = -u(\nu)d\nu$.

Radiant intensity per unit wavelength per solid angle is $I(\lambda) = \frac{c}{4}u(\lambda)$ and this relation just related the intensity to energy density for this kind of radiation. This is the formula followed in the reference book (Serway) so I will be sticking to this and also you would have received a handwritten note with it's derivation. The same relation holds for intensity per unit frequency.

Q3

(a) We wish to maximize $u(\lambda, T)$ at fixed T so essentially make $\frac{du}{d\lambda} = 0$ and $\frac{d^2u}{d\lambda^2} < 0$. Now we can calculate the following:

$$\frac{du}{d\lambda} = \frac{8\pi hc}{\lambda^6} \frac{1}{\exp(\frac{hc}{\lambda k_B T}) - 1} \left(\frac{\frac{hc}{\lambda k_B T} \exp(\frac{hc}{\lambda k_B T})}{\exp(\frac{hc}{\lambda k_B T}) - 1} - 5 \right)$$

To solve this equal to zero one can use a generic scientific calculator (the CASIO fx-991EX will be the best choice) where one can use the table mode set $f(x) = \frac{xe^x}{e^x - 1}$ and $g(x) = 5$ and you will get $x \approx 4.97$ (you can also sort of guess it from the form of equation here). So we get that $\frac{hc}{\lambda_{max} k_B T} = 4.97$ and clearly from the graph of $u(\lambda)$ versus λ there is only one maxima so this must be it so we get

$$\lambda_{max} = \frac{hc}{4.97 k_B T}$$

(b) Simply substitute the expression we get

$$u(\lambda_{max}, T) = \frac{8\pi T^5 hc}{\alpha^5} \frac{1}{\exp(\frac{hc}{\alpha k_B T}) - 1}$$

Q5

(a) Rayleigh Jean's law holds correct in the limit of small frequency so using that on planck's law and we get $\exp(\frac{h\nu}{k_B T}) - 1 \approx \frac{h\nu}{k_B T}$ so just substitute that in planck's law and we have

$$U(\nu, T)d\nu = \frac{8\pi\nu^2}{c^3} kT d\nu$$

(b) So essentially we have $u(\nu_0) = 0.1 \times U(\nu_0)$ where u is the measured value and U is the value from rayleigh jeans, so you just gotta solve this and cancel the useless terms and you should get

$$\frac{\frac{h\nu}{k_B T}}{\exp(\frac{h\nu}{k_B T}) - 1} = 0.1$$

(c) Again using a scientific calculator we get $\frac{h\nu}{k_B T} \approx 3.61$

Q7

Stefan's law is used to describe the total energy emitted per unit area of the blackbody at a certain temperature and that would be the integral over all wavelengths of the intensity $I(\lambda, T)$ so we get

$$e_{total} = \int_0^\infty I(\lambda, T) d\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp(\frac{hc}{\lambda k_B T}) - 1} d\lambda$$

Now substitute $x = \frac{hc}{\lambda k_B T}$ and this transforms to

$$e_{total} = \frac{2\pi k_B^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

Now you may not be familiar with this integral but for now while this should have been mentioned in the question take $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$ using this we finally get:

$$e_{total} = \frac{2\pi^5 k_B^4}{15c^2 h^3} T^4$$

Taking $\sigma = \frac{2\pi^5 k_B^4}{15c^2 h^3}$ we will get the good old stefan boltzmann equation:

$$e_{total} = \sigma T^4$$

And you can remove the value of σ and it would come to be $5.67 \times 10^{-8} W m^{-2} K^{-4}$ which you probably remember from your JEE days. For those interested in seeing how one can evaluate the integral normally refer [here](#). You will learn this if you take any complex analysis course in the future.

Tutorial 2

Useful formulae for this tut: Compton's effect

$$\lambda' - \lambda = \lambda_c(1 - \cos(\theta)) \quad (3)$$

$$\lambda_c = \frac{h}{m_0c} \quad (4)$$

$$\cot(\theta/2) = \left(1 + \frac{\lambda_c}{\lambda}\right) \tan(\phi) \quad (5)$$

Apart from this using momentum and energy conservation should easily get you through this topic.

Q1

We know that $\lambda' - \lambda = \lambda_c(1 - \cos\theta)$ and $\theta = 90^\circ$ and we are given that the wavelength of scattered photon is twice of that of the incoming photon so.

$$\lambda' - \lambda = \lambda_c, \quad \lambda' = 2\lambda$$

So we have $\lambda = \lambda_c$ and from that we get $\nu = c/\lambda = 1.23 \times 10^{20}\text{Hz}$.

For calculating angle of electron we know it has two perpendicular components of momentum one which is twice of other (why?). From that we get that it makes an angle of $\arctan(0.5)$ with horizontal and that is approx 26.6°

Q4

(a) Increase in wavelength is given by $\frac{h}{m_0c}(1 - \cos\theta)$ so here we have that value for 45° as $7 \times 10^{-14}m$ so that gives us

$$\frac{h}{m_0c} \left(1 - \frac{1}{\sqrt{2}}\right) = 7 \times 10^{-14}$$

Which on solving gives $m_0 \approx 9.23 \times 10^{-30}$.

(b) We know wavelength of the scattered photon in second case at $\theta = 60^\circ$ is $9.9 \times 10^{-12}m$ so if incoming wavelength is λ for the first experiment (making it 2λ for the second one since $E_2 = E_1/2$) then we have

$$9.9 \times 10^{-12} - \lambda_c(1 - \cos(60^\circ)) = 2\lambda$$

Solving this we get $\lambda = 4.89 \times 10^{-12}m$.

Q6

So first off the question in the actual tut is framed somewhat incorrectly the actual question is as follows: *Consider Compton Scattering. Show that if the angle of scattering increases beyond a certain value of θ_0 , the scattered photon will never have energy larger than $2m_0c^2$, irrespective of energy of incident photon. Find the value of θ_0* (Source: last year's tut).

For maximum energy of scattered photon we need λ' minimized

$$\lambda' - \lambda = \lambda_c(1 - \cos(\theta))$$

To minimize λ' we can take $\lambda = 0$ in theory.

$$\lambda'_{min} = \lambda_c(1 - \cos(\theta))$$

We have been given that the maximum energy for this situation is $2m_0c^2$ as long as $\theta > \theta_0$ so this gives

$$\frac{hc}{\lambda'_{min}} = 2m_0c^2$$

This gives us that $\theta_0 = 60^\circ$ making θ smaller increases the energy.

Q7

(a)

$$\lambda_2 - \lambda_1 = \lambda_c(1 - \cos(\theta))$$

$$\lambda_3 - \lambda_2 = \lambda_c(1 - \sin(\theta))$$

We add this up and write $\sin(\theta) + \cos(\theta) = \sqrt{2} \cos(\theta - \frac{\pi}{4})$ and this gives $\theta \approx 59.83^\circ$ since it is given to be greater than $\pi/4$.

(b) You will get from the given λ_2 value that $\lambda_1 = \lambda_2 - \lambda_c/2 \approx 65.57 \times 10^{-12}m$. Use 5 after this and you will get $\phi \approx 60^\circ$. Do derive equation 5 using energy conservation and check for yourselves.

Tutorial 3 & 4

Useful formulae for this tut:

De broglie wavelength:

$$\lambda = \frac{h}{p}$$

Einstein solid expression for internal energy and einstein temperature:

$$U = \frac{3Nh\nu}{e^{\frac{h\nu}{k_B T}} - 1}$$

$$\Theta_E = \frac{h\nu_E}{k_B}$$

Specific Heat

Q3

So in graphite which is a solid we only have vibrational degrees of freedom. We have a total of three independent directions for vibration each which will contribute a R to the C_v . Now we are given that $h\omega_{x,y} \gg 300k_B \gg h\omega_z$ so that would mean that we have enough energy for vibration along z direction but not for other directions hence we get that $C_v = R$.

Q4

We will start at $5R/2$ since its given that the rotational excitation energy is negligible. We use $k_B = 8.62 \times 10^{-5} eV \cdot K^{-1}$. For the first vibrational mode we will need $k_B T = E$ so this gives us $T \approx 139.21K$. The next transition will be at $4T \approx 556.84$ where we get a $+R$ to C_v and then $6T \approx 835.27$ where we get a $+2R$ to C_v since two degenerate energy levels of vibration. Refer figure below.

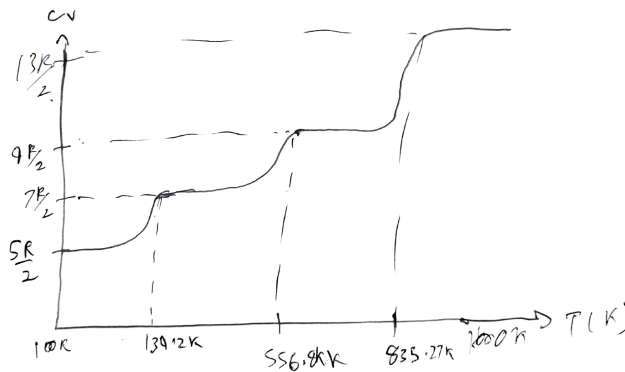


Figure 1: C_v vs. T for CO_2

De Broglie

Q2

So we have quantized angular momentum for a hydrogen atom

$$mvr = \frac{nh}{2\pi}$$

$$2\pi r = \frac{nh}{mv}$$

Now if we say that electron has de broglie wavelength of λ then we essentially have

$$2\pi r = n\lambda$$

Note that n is an integer so we essentially have that an integer times wavelength equals the circumference of our orbit which is exactly what happens in a standing wave where essentially your wavelength repeats for an integer number of times over a given length. The wave would look odd but would be a standing wave over the circumference and is called a standing circular wave.

Q4

For a photon $p = E/c$ so we have

$$\lambda_{ph} = hc/E = 0.2475 \text{ nm} \approx 0.25 \text{ nm}$$

For the other two just use $p = \sqrt{2mE}$ so we have

$$\lambda_e = \frac{hc}{\sqrt{2 \times 500 \text{ keV} \times 5 \text{ keV}}} \approx 0.0175 \text{ nm}$$

$$\lambda_n = \frac{hc}{\sqrt{2 \times 1000 \text{ MeV} \times 5 \text{ keV}}} \approx 3.92 \times 10^{-4} \text{ nm}$$

Tutorial 5

Useful formulae for this tut: phase and group velocities

$$v_p = \frac{\omega}{k} \quad (6)$$

$$v_g = \frac{d\omega}{dk} \quad (7)$$

$$v_g = v_p + k \frac{dv_p}{dk} \quad (8)$$

In three dimensions we write

$$\vec{v}_g = \nabla_k \omega = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right) \quad (9)$$

Q1

(a) Resultant wave is

$$y = y_1 + y_2 = 0.004 \cos(7.8x - 390t) \cos(0.2x - 10t)$$

$$v_p = \frac{\omega + \Delta\omega}{k + \Delta k} = 50 \text{ m/s}$$

$$v_p = \frac{\Delta\omega}{\Delta k} = 50 \text{ m/s}$$

(b) The group wave is the $\cos(0.2x - 10t)$ part and so clearly $\Delta x = 5\pi$ m. We can also see that $\Delta k = 8 - 7.6 = 0.4 \text{ m}^{-1}$ so we get that $\Delta x \Delta k = 2\pi$

Q4

Since we know that $k = 2\pi/\lambda$ we get that

$$v_p = \frac{c}{A + Bk^2}$$

From this we get

$$\omega = kv_p = \frac{ck}{A + Bk^2}$$

(a)

$$v_g = \frac{d\omega}{dk} = \frac{(A - Bk^2)}{(A + Bk^2)^2}$$

(b) Given $\lambda = 0.4 \text{ }\mu\text{m}$. Using this and given constants we get $Bk^2 = 0.0625$. Hence we get $v_p = c/(1.7+0.0625) \approx 0.567c$ and $v_g = 0.527c$. The v_g is more important since it gives the velocity of the group wave and if we treat the light as photons (essentially the wave packets) then the velocity of the wave packets in this dispersive medium will be that value. Also phase velocity cannot actually be used for "sending" information since it has the same value throughout space.

Q5

Given

$$-\omega^2 m + 2\beta_x(1 - \cos(k_x a_x)) + 2\beta_y(1 - \cos(k_y a_y)) = 0$$

From this expression we get

$$\frac{\partial \omega}{\partial k_x} = \frac{a_x \beta_x \sin(k_x a_x)}{m\omega}$$

$$\frac{\partial \omega}{\partial k_y} = \frac{a_y \beta_y \sin(k_y a_y)}{m\omega}$$

So we get

$$\vec{v}_g = \frac{\partial \omega}{\partial k_x} \hat{i} + \frac{\partial \omega}{\partial k_y} \hat{j} = \frac{a_x \beta_x \sin(k_x a_x) \hat{i} + a_y \beta_y \sin(k_y a_y) \hat{j}}{\sqrt{m(2\beta_x(1 - \cos(k_x a_x)) + 2\beta_y(1 - \cos(k_y a_y)))}}$$

The angle it makes with x -axis would be

$$\theta = \arctan \left(\frac{a_x \beta_x \sin(k_x a_x)}{a_y \beta_y \sin(k_y a_y)} \right)$$

Tutorial 6

Useful formulae for this tut: Fourier transform and uncertainty principle

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-\iota kx} dx \tag{10}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{\iota kx} dk \tag{11}$$

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{12}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{13}$$

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle| \tag{14}$$

Equation 14 is essentially cyclic and not in portion but ahead you may get an idea of why this relation exists.

Fourier Transform

Q1

(a)

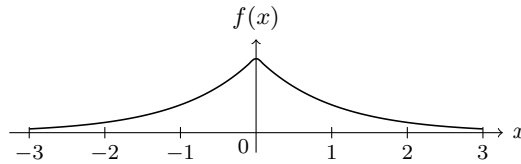


Figure 2: $f(x) = e^{-\alpha|x|}$ versus x

(b) $f(x)$ will attain half it's max value at $x = 0$ which is of value 1 so for half the max we get $x = \pm \ln(2)/\alpha$.

(c) Here whether you use $+\iota kx$ or $-\iota kx$ you will get the same answer and that would be

$$g(k) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + k^2}$$

I have taken the $\frac{1}{\sqrt{2\pi}}$ also for getting this answer (d)

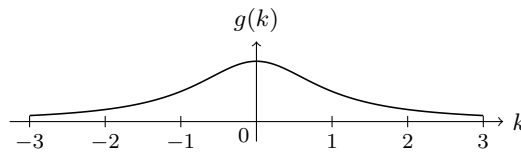


Figure 3: $g(k) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + k^2}$ versus k

(e) $g(k)$ reaches it's maximum value at $k = 0$ which is $\frac{1}{\alpha} \sqrt{\frac{2}{\pi}}$ and so for reaching half it's max value we will have $k = \pm \alpha$.

(f) We have from (b) and (e) that $\Delta x = 2 \ln(2)/\alpha$ and $\Delta k = 2\alpha$ so we have $\Delta x \Delta k = 4 \ln(2)$

Q4

$A(k)$ over here essentially represents weights of each $\cos(kx - \omega t)$ for forming $y(x, t)$. From the given $A(k)$ we get that

$$y(x, t) = \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} A \cos(kx - \omega t) dx = \frac{A}{x} \sin\left(\frac{\Delta kx}{2}\right) \cos(k_0x - \omega t)$$

The envelope part would be defined by $y_g = \frac{A}{x} \sin\left(\frac{\Delta kx}{2}\right)$ since $\cos(k_0x - \omega t)$ is the oscillatory part. The minimum for the envelope would be where $|y_g|$ is minimised since that is the amplitude for the phase wave

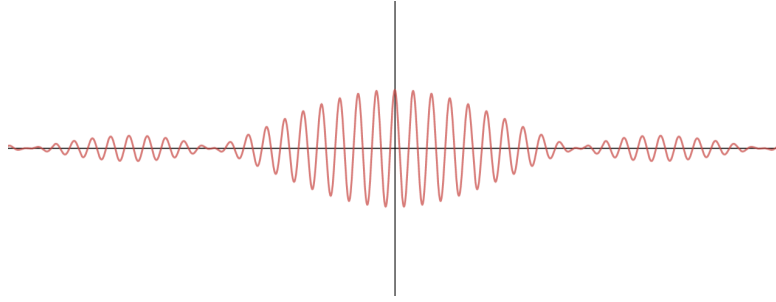


Figure 4: $y(x,t)$ versus x at $t = 0$

and that defines the actual envelope. This is minimised at $y_g = 0$ and we can clearly see that happens at the following for the first time

$$\frac{\Delta kx}{2} = \pi$$

Hence for getting the first minimum you will get $x = \frac{2\pi}{\Delta k}$ and the central maximum is clearly $x = 0$ so using this we get

$$\Delta x = \frac{2\pi}{\Delta k}$$

And so we get that $\Delta x \Delta k = 2\pi$ which is clearly greater than 0.5 so we have verified the uncertainty principle.

Uncertainty Principle

Q8

(a) Diameter of 10^{-14} m means that $\Delta x = 10^{-14}/2$ which is error in the radial coordinate of electron. Using uncertainty principle we get that the minimum error in radial momentum is $\Delta p = 1.05 \times 10^{-20}$ kg m s⁻¹ or we can say that $pc = 19.69$ MeV. We can take average of momentum to be zero since it's in a orbit and we know that $m_0c^2 = 0.51$ MeV so we get

$$E = \sqrt{p^2c^2 + m_0^2c^4} \approx 19.7 \text{ MeV}$$

This gives us $KE = 19.7 - 0.51 = 19.19$ MeV which is much larger than the observed kinetic energies which are in order of 1 MeV for an electron in an atom hence disproving existence of the electron in the nucleus.

Similarly for the proton the momentum will be the same and $m_c^2 = 938.37$ MeV so

$$KE = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 = 0.21 \text{ MeV.}$$

(b) For a harmonic oscillator we have the following

$$\langle E \rangle = \frac{1}{2}k\langle x^2 \rangle + \frac{\langle p_x^2 \rangle}{2m}$$

But we know that $\langle x \rangle = \langle p_x \rangle = 0$ so then we have the following using AM-GM inequality

$$\langle E \rangle = \frac{1}{2}k(\Delta x)^2 + \frac{(\Delta p)^2}{2m} \geq \sqrt{\frac{k}{m}(\Delta x)^2(\Delta p)^2}$$

But we know that $\frac{k}{m} = \omega^2$ which is the harmonic oscillator frequency. Using the uncertainty principle we get another lower limit as

$$\langle E \rangle \geq \sqrt{\frac{k}{m}(\Delta x \Delta p)} \geq \frac{1}{2}\hbar\omega = \frac{1}{2}hf$$

So $\langle E \rangle_{min} = \frac{1}{2}hf$ which is ground state energy. Note that inequality becomes equality for a Gaussian wave packet and on solving Schrodinger equation for a harmonic oscillator, the ground state comes out to be Gaussian (you will see this come ahead).

(c) Let the radius be some a then we have $\Delta x = a$ and we will have $\Delta p \approx \hbar/a$. We will take average of p to be zero since it's an orbit so then $p \approx \Delta p$. The expression of energy will then be

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 a} = \frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\epsilon_0 a}$$

To minimise this we will make $\frac{\partial E}{\partial a} = 0$ and you can see that this will happen at

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.529 \times 10^{-10} \text{ m}$$

Q10

(a) For the collision we will conserve the momentum first. Since photon is emitted with energy E that means $p = E/c$ of the photon. This means the atom would get the exact same momentum but in opposite direction to conserve momentum so we get that it will move with velocity of $v = \frac{E}{mc}$ this gives recoil energy of

$$E_{recoil} = mv^2/2 = \frac{E^2}{2mc^2}$$

(b) We have been given $\Delta t = 10^{-8}$ s so we can use the uncertainty relation as

$$\Delta E \geq \frac{\hbar}{2\Delta t}$$

So we get the minimum $\Delta E = 0.53 \times 10^{-26}$ J = 3.3×10^{-8} eV from here and so the spectral line width would be $2\Delta E$ since energy lies between $E - \Delta E$ and $E + \Delta E$ which is 6.6×10^{-8} eV.

(c) We essentially want $E_{recoil} = 2\Delta E$. So we will have

$$\frac{E^2}{2mc^2} = 6.6 \times 10^{-8} \text{ eV}$$

This gives us $E = 111.35$ eV.

(d) So first let's discuss what resonant absorption theory is (this is not part of your portion exactly as in such a question will not come in your exams). When a photon causes a energy level shift by E_0 in the atom, the energy of photon will be greater than this since the atom will also have some kinetic energy given to it. Similarly if we have an excited state go down by E_0 , the emitted photon would have energy less than E_0 since the atom will have some kinetic energy as recoil as the photon is emitted. What we want for resonant absorption is essentially something like a photon enters, excites one atom which then comes back to ground state and that released photon excites another atom and so on. This can only happen if that additional kinetic energy lies within that error range of the spectral line and so we want the recoil energy to be same order as the spectral line width so that way the extra energy required in recoil compared to actual line width will be adjusted in the error of the actual E value. You can read more about it [here](#).

Q12

(a) We have been given that it is a non relativistic electron so kinetic energy is simply $\frac{p^2}{2m}$. Taking $\Delta x = a$ we get minimum $\Delta p = \frac{\hbar}{2a}$. Taking $p = \Delta p$ we get that

$$KE = \frac{p^2}{2m} = \frac{\hbar^2}{8ma}$$

(b) Given that $\Delta x = \frac{\hbar}{mv}$ so we can use the uncertainty relation and we will see that $\Delta v = \frac{v}{4\pi}$ hence they have the same order which is mostly what the question meant here.

(c) We know that $v = \sqrt{2E/m} = 8.39 \times 10^6$ m/s hence $\Delta v = \frac{v}{4\pi} = 6.68 \times 10^5$ m/s using the result of (b)

(d) As it enters the hole it will undergo diffraction. For the hole we have $r = 10^{-6}$ so $\Delta x = 2r = 2 \times 10^{-6}$ m. Using this we have $\Delta p_x = \hbar/(2\Delta x) = 2.6 \times 10^{-27}$ kg m/s. Now note that this is an error in momentum along the component along the plane of the hole. Since we have $E = 0.2$ keV we know that $p = \sqrt{2mE} = 7.6 \times 10^{-24}$ kg m/s so error in the component of this along plane of hole is very small in comparison to this so under that assumption we can conclude

$$\theta \approx \frac{p_x}{p}$$

This gives us that

$$\Delta\theta \approx \frac{\Delta p_x}{p} = 3.4 \times 10^{-4} \text{ rad}$$

Tutorial 7

Useful formulae for this tut: Schrödinger's equations. Here \hat{H} is the hamiltonian of the system. It's essentially $-\frac{\hbar^2}{2m}\nabla^2 + V$ for a single particle.

$$\hat{H}\psi(x, t) = E\psi(x, t) \text{ the TISE} \quad (15)$$

$$\iota\hbar\frac{\partial\psi(x, t)}{\partial t} = \hat{H}\psi(x, t) \text{ the TDSE} \quad (16)$$

Q1

We have been given that

$$\hat{H}\phi_n(x) = E_n\phi_n(x)$$

So for the given $\psi(x, t) = \sum_n C_n\phi_n(x)\exp(-\iota E_n t/\hbar)$ we can see that

$$\iota\hbar\frac{\partial\psi(x, t)}{\partial t} = \sum_n C_n E_n \phi_n(x) \exp(-\iota E_n t/\hbar)$$

But $\hat{H}\phi_n(x) = E_n\phi_n(x)$ so we get that

$$\iota\hbar\frac{\partial\psi(x, t)}{\partial t} = \sum_n C_n E_n \phi_n(x) \exp(-\iota E_n t/\hbar) = \sum_n C_n \hat{H}\phi_n(x) \exp(-\iota E_n t/\hbar) = \hat{H}\psi(x, t)$$

Hence it satisfies the TDSE. Let's assume that TISE satisfies for $\psi(x, t)$, we can see that then we should have

$$\hat{H}\psi(x, 0) = \sum_n C_n E_n \phi_n(x) = E\psi(x, t)$$

This gives us

$$\sum_n C_n (E_n - E)\phi_n(x) = 0$$

If we have $\phi_n(x)$ as distinct solutions, they must have unique eigen values and must form an orthogonal space since \hat{H} is a hermitian operator ([you can find a good explanation of why this is true here](#)) hence this is not possible to be satisfied hence we have that $\psi(x, 0)$ doesn't satisfy the TISE.

Q5

$$\begin{aligned} \hat{G}\phi(x) &= k\phi(x) \\ \iota\hbar\frac{d\phi(x)}{dx} &= (k - Ax)\phi(x) \end{aligned}$$

Solving this gives us

$$\phi(x) = \exp\left(\frac{-\iota}{\hbar}\left(kx - \frac{Ax^2}{2} + c\right)\right)$$

We have been given that $\phi(x)$ is even however that is only possible if $k = 0$ so we only have one eigen value and that is $k = 0$.

Q6

The probability of finding the particle in an interval x to $x + dx$ is $\phi^*(x)\phi(x)dx$ so we are given that at $x = 2a$ this interval had 100 particles measured in it so

$$A^2 \exp(-2(2a)^2/a^2)dx = \frac{100}{N}$$

$$NA^2 dx = 100e^8$$

So at $x = a$ we have the number of measurements as

$$N(A^2 \exp(-2)dx) = 100e^6$$

Q7

This is not normalized so to normalize we will have to rewrite it as

$$\psi(x) = \frac{1}{\sqrt{10}}\psi_1(x) + \frac{3}{\sqrt{10}}\psi_2(x)$$

Now to observe P_1 we need the wavefunction to collapse to $\psi_1(x)$ and the probability of that is proportional to the square of it's coefficient and since we have normalized it, the probability of observing P_1 is simply $\frac{1}{10}$.

Tutorial 8

Useful formulae for this tut:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \quad (17)$$

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad (18)$$

Q5

We have been given the wave function as $\Psi = Ae^{i(kx-\omega t)}$. We need expectation value of square of momentum squared which is essentially $\langle p^4 \rangle$

$$\langle p^4 \rangle = \frac{\int_{-\infty}^{\infty} \Psi^* \hbar^4 \frac{\partial^4}{\partial x^4} \Psi dx}{\int_{-\infty}^{\infty} \Psi^* \Psi dx}$$

But we know that $\frac{\partial \Psi}{\partial x} = ik\Psi$ and so $\frac{\partial^4 \Psi}{\partial x^4} = k^4\Psi$ and using this we get

$$\langle p^4 \rangle = \frac{\int_{-\infty}^{\infty} \Psi^* \hbar^4 k^4 \Psi dx}{\int_{-\infty}^{\infty} \Psi^* \Psi dx} = \hbar^4 k^4 \frac{\int_{-\infty}^{\infty} \Psi^* \Psi dx}{\int_{-\infty}^{\infty} \Psi^* \Psi dx} = \hbar^4 k^4$$

The conclusion we can draw from this solution is that it is composed of only one eigenfunction of the momentum operator and hence only can give one momentum value on measurement and so we would essentially have $\langle p^n \rangle = \langle p \rangle^n$ since our statistic has only one value which will always be attained on measurement.

Tutorial 9

Useful formulae for this tut: Particle in an infinite well from $x = 0$ to $x = L$, eigen functions and energy values:

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad (19)$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad (20)$$

Q1

We know that E_n is proportional to n^2 and so the limit of high energy just means very large n . Now assuming that the state is an eigenfunction we can write probability as

$$P = \int_a^{a+b} |\phi_n(x)|^2 dx = \frac{2}{L} \int_a^{a+b} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_a^{a+b} \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$

On evaluating the integral we get

$$P = \frac{b}{L} - \frac{x}{2n\pi x} \left(\sin\left(\frac{2n\pi(a+b)}{L}\right) - \sin\left(\frac{2n\pi a}{L}\right) \right)$$

For large n the second term would go to zero and so $P \approx \frac{b}{L}$.

Q3

The clearest way is to just plot how $|\phi_3(x)|^2$ versus x looks like The probability would be in proportion to the

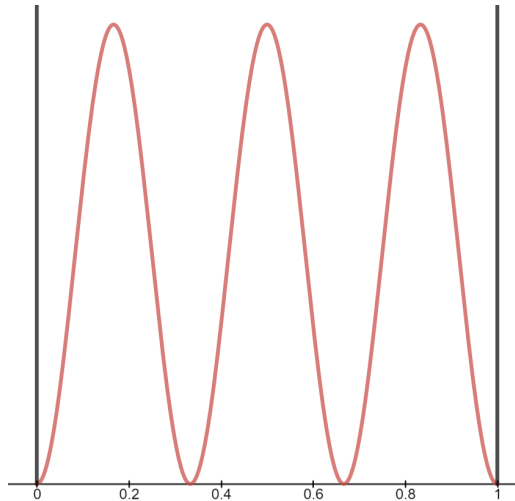


Figure 5: $|\phi_3(x)|^2$ versus x

are under the graph from 0 to $L/6$ and it is very clearly equal to $1/6$ from this. To verify we can check that the following holds

$$P = \int_0^{L/6} \frac{2}{L} \sin^2\left(\frac{3\pi x}{L}\right) dx = \frac{1}{6}$$

Q7

(a) First we represent $\psi(x, 0)$ in terms of the eigen functions

$$\psi(x, 0) = \frac{2}{\sqrt{L}} \sin\left(\frac{3\pi x}{2L}\right) \cos\left(\frac{\pi x}{2L}\right) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \right) = \frac{1}{\sqrt{2}} (\phi_1(x) + \phi_2(x))$$

Hence solving the TDSE each eigen function would have a $\exp(-\iota Et/\hbar)$ multiplied to it of that eigen functions energy value. So we would write it as

$$\psi(x, t) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \exp\left(\frac{-\iota E_1 t}{\hbar}\right) + \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \exp\left(\frac{-4\iota E_1 t}{\hbar}\right) \right)$$

Where $E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$

(b) At any given time the probability for finding particle between $L/4$ and $L/2$ would be

$$P = \int_{L/4}^{L/2} |\psi(x, t)|^2 dx$$

which we can write as

$$P = \int_{L/4}^{L/2} \frac{1}{2} \left(\phi_1(x) \exp\left(\frac{iE_1 t}{\hbar}\right) + \phi_2(x) \exp\left(\frac{4iE_1 t}{\hbar}\right) \right) \left(\phi_1(x) \exp\left(\frac{-iE_1 t}{\hbar}\right) + \phi_2(x) \exp\left(\frac{-4iE_1 t}{\hbar}\right) \right) dx$$

On simplification this becomes into

$$P = \frac{1}{2} \int_{L/4}^{L/2} (\phi_1(x)^2 + \phi_2(x)^2 + 2\phi_1(x)\phi_2(x) \cos(3E_1 t/\hbar)) dx$$

And to write more properly this would be

$$P = \frac{1}{L} \int_{L/4}^{L/2} \left(\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) \cos(3E_1 t/\hbar) \right) dx$$

On evaluating this we get

$$P = \frac{1}{4} + \frac{1}{4\pi} + \frac{4 - \sqrt{2}}{3\pi} \cos\left(\frac{3\pi^2 \hbar t}{2mL^2}\right)$$

Q8

(a) We essentially have some $E_n = 1\mu\text{J}$ and so we just do

$$\frac{n^2 \pi^2 \hbar^2}{2mL^2} = 10^{-6}$$

On putting the value we will get $n \approx 1.35 \times 10^{20}$

(b) We can clearly see that n is a pretty darn large number so using the logic from **Q1** we can directly conclude that this would be 0.1 probability.

(c) Since n is large the difference between the next level would be $\Delta E = E_{n+1} - E_n = ((n+1)^2 - n^2)E_1 \approx 2nE_1$ and so we find that value to be

$$\Delta E \approx \frac{n\pi^2 \hbar^2}{mL^2} = 1.47 \times 10^{-26} \text{ J}$$

At room temperature $k_B T \approx 4.14 \times 10^{-21} \text{ J}$ which shows that at room temperature the spectra of energy levels look quasi-continuous for this dust particle.

Tutorial 10

Q1

Since these are bound states we get the condition that $E < V_2$ and $E < V_1$ (why?). So we now define three regions, I is $x < 0$, II is $0 < x < L$ and III is $L < x$. Our wavefunction would take the following values as per region

$$\begin{aligned}\psi_I(x) &= Ae^{k_1x} \\ \psi_{II}(x) &= B \sin(k_2x) + C \cos(k_2x) \\ \psi_{III}(x) &= De^{-k_3x}\end{aligned}$$

Here $k_1 = \sqrt{2m(V_1 - E)/\hbar^2}$, $k_2 = \sqrt{2m(E)/\hbar^2}$, $k_3 = \sqrt{2m(V_2 - E)/\hbar^2}$ Now we apply the magical boundary conditions that ψ is continuously differentiable. So we get the following equations for between I and II

$$A = C$$

$$Ak_1 = Bk_2$$

And

$$\begin{aligned}B \sin(k_2L) + C \cos(k_2L) &= De^{-k_3L} \\ -Ck_2 \sin(k_2L) + Bk_2 \cos(k_2L) &= -Dk_3e^{-k_3L}\end{aligned}$$

So using these four equations we can write the following equation after elimination (just divide fourth equation by third and then substitute everything in terms of C and after rearranging you will get this)

$$\tan(k_2L) = \frac{k_3k_2 + k_1k_2}{k_2^2 - k_3k_1}$$

Now if we apply $V_1 \rightarrow \infty$ then k_1 also will go to infinity and we will then get

$$\tan(k_2L) = \frac{-k_2}{k_3}$$

And you can see that this is just the anti-symmetric solutions of a symmetric finite well of length $2L$ centered at $x = 0$ and that makes sense if you see that $x = 0$ should be a node for the wavefunction.

Q2

(a) Let us again break this into three regions, I is $-2L < x < -L$, II is $-L < x < L$ and III is $L < x < 2L$. Let us define $k = \sqrt{2mE/\hbar^2}$. Using this we can see that

$$\begin{aligned}\psi_I(x) &= A \sin(kx) + B \cos(kx) \\ \psi_{II}(x) &= C \\ \psi_{III}(x) &= D \sin(kx) + E \cos(kx)\end{aligned}$$

Using boundary conditions of wavefunction being continuously differentiable in $(-2L, 2L)$ and being zero at $\pm 2L$ due to infinite potential we get

$$\begin{aligned}A \sin(2kL) &= B \cos(2kL) \\ -D \sin(2kL) &= E \cos(2kL) \\ -A \sin(kL) + B \cos(kL) &= D \sin(kL) + E \cos(kL) = C \\ A \cos(kL) + B \sin(kL) &= D \cos(kL) - E \sin(kL) = 0\end{aligned}$$

From this we can use the first and fourth equation to get the quantization condition as $\tan(2kL) = -\cot(kL)$ which can be written as

$$kL = \frac{(2n+1)\pi}{2}$$

(b) Using the previous equation we can see that $B = E = 0$ and $|A| = |D| = C$. Hence we can find what value we should get on finding the norm of wavefunction and then apply normalization condition.

$$\int_{-2L}^{2L} |\psi(x)|^2 dx = 2LC^2 + 2C^2 \int_L^{2L} \sin^2\left(\frac{(2n+1)\pi x}{2L}\right) dx = 3LC^2$$

So we get that $C = \sqrt{\frac{1}{3L}}$

(c) The condition of V_0 can be obtained by using the allowed kL values and so

$$V_0 = \frac{(2n+1)^2 \pi^2 \hbar^2}{8mL^2}$$

This gives us the two lowest values as $V_0 = 9.34$ eV and $V_0 = 84.08$ eV.

(d) Trivially we can see that $\langle x \rangle = 0$ since ground state is symmetric. You can check that $\langle p \rangle = 0$ by integration.

$$\langle \hat{x}^2 \rangle = 2 \int_L^{2L} \frac{x^2}{3L} \sin^2\left(\frac{\pi x}{2L}\right) dx + \int_{-L}^L \frac{x^2}{3L} dx = L^2 \left(1 - \frac{2}{\pi^2}\right)$$

$$\langle \hat{p}^2 \rangle = 2 \frac{\hbar^2 \pi^2}{4L^2} \int_L^{2L} \frac{1}{3L} \sin^2\left(\frac{\pi x}{2L}\right) dx = \frac{\hbar^2 \pi^2}{12L^2}$$

From this we can write $\Delta x \Delta p = \sqrt{\langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle}$ and we can evaluate that to get

$$\Delta x \Delta p = \frac{\hbar}{2} \sqrt{\frac{\pi^2 - 2}{3}} > \frac{\hbar}{2}$$

So we have verified the uncertainty principle.

Q4

If there is an infinite wall at $x = 0$ then it is necessary that the wavefunction must also have a node at that point. Now we know that the lowest energy bound state would be the ground state which would be even and not zero at $x = 0$. Only the $(2n+1)$ th excited state would have a node at $x = 0$ and so the only allowed energies are 7 eV, 17 eV and 24 eV.

Tutorial 11

Q2

In the region where $E < V$ we know that due to boundary conditions, the wavefunction $\psi(x) = A \exp(-\alpha x)$ and from the TISE we can say that $\alpha = \sqrt{\frac{2m(V-E)}{\hbar^2}}$. By putting in values we can see that $\alpha \approx 1.03 \times 10^{10} \text{ m}^{-1}$. The final answer would be where the value of ψ is $1/\sqrt{2}$ times that of what it is at $x = 0$ and that would be at $x = \frac{\ln 2}{2\alpha} \approx 3.37 \times 10^{-11} \text{ m}$.

Q3

(a) Let us divide this into three regions, I is $x < 0$, II is $0 < x < L$ and III is $x > L$. The potential $V = E$ in region II and zero elsewhere. So we can solve the TISE and obtain the following. $k = \sqrt{\frac{2mE}{\hbar^2}}$ here

$$\psi_I(x) = Ae^{\iota kx} + Be^{-\iota kx}$$

$$\psi_{II}(x) = Cx + D$$

$$\psi_{III}(x) = Fe^{\iota kx}$$

We now apply the good old boundary conditions of being continuously differentiable.

$$A + B = D$$

$$A - B = C \frac{-\iota}{k}$$

$$F = (CL + D)e^{-\iota kL}$$

$$F = Ce^{-\iota kL} \frac{-\iota}{k}$$

With some manipulation we can get the following expression

$$\frac{F}{A} = \frac{2e^{-\iota kL}}{2 - \iota kL}$$

This gives the transmission coefficient to be

$$T = \frac{|F|^2}{|A|^2} = \frac{2\hbar^2}{2\hbar^2 + mEL^2}$$

(b) Using some manipulation we can similarly find the following

$$\frac{B}{A} = \frac{kL}{-2\iota - kL}$$

We can remove the reflection coefficient as

$$R = \frac{|B|^2}{|A|^2} = \frac{mEL^2}{2\hbar^2 + mEL^2}$$

For $R = 1/2$ we can see that we need $kL = 2$ from the expression of R hence we have

$$L = \frac{\lambda}{\pi}$$

Q4

The claim is not entirely correct for the wavefunction in region I. Since there is another interface where reflection can happen, we have to also consider e^{k_1x} in this region and then accordingly solve the boundary conditions taking $\psi_I(x) = Ae^{-k_1x} + Ce^{k_1x}$ and $\psi_{II}(x) = Be^{-k_2x}$. Why can't we just take $C = 0$? Well if we do, then the boundary conditions at $x = L$ will be quite problematic since we would have $A = Be^{k_1L - k_2L}$ and also $A = \frac{k_1}{k_2} Be^{k_1L - k_2L}$. Hence we would actually have some non zero value for C .

Q5

(a) As usual we will break this into three regions, I is $x < 0$, II is $0 < x < d$ and III is $d < x$. Let us declare $k_1 = \sqrt{2m(9V_0)/\hbar^2}$, $k_2 = \sqrt{2m(4V_0)/\hbar^2}$ and $k_3 = \sqrt{2m(9-n)V_0/\hbar^2}$.

$$\psi_I = Ae^{\iota k_1 x} + Be^{-\iota k_1 x}$$

$$\psi_{II} = Ce^{\iota k_2 x} + De^{-\iota k_2 x}$$

$$\psi_{III} = Fe^{\iota k_3 x}$$

Now we apply boundary conditions

$$A + B = C + D$$

$$A - B = (C - D) \frac{k_2}{k_1}$$

$$Ce^{\iota k_2 d} + De^{-\iota k_2 d} = Fe^{\iota k_3 d}$$

$$Ce^{\iota k_2 d} - De^{-\iota k_2 d} = Fe^{\iota k_3 d} \frac{k_3}{k_2}$$

From given value of $d = \pi\hbar/\sqrt{8mV_0}$, we have $k_2 d = \pi$ and so

$$\frac{C + D}{C - D} = \frac{k_2}{k_3} = \frac{A + B}{A - B} \times \frac{2}{3}$$

This gives us

$$\frac{1 + \frac{B}{A}}{1 - \frac{B}{A}} = \sqrt{\frac{9}{9-n}}$$

We know that reflection coefficient would have to be 0.25 and that gives us $B/A = 0.5e^{\iota\phi}$ and since LHS has to be real in above equation that forces us to choose ± 0.5 and so we get

$$\sqrt{\frac{9}{9-n}} = 3 \text{ or } \frac{1}{3}$$

and so $n = 8$ or -72 .

(b) We can calculate and find that for $n = 8$,

$$\psi_I = Ae^{\iota k_1 x} + Ae^{-\iota k_1 x}/2$$

$$\psi_{II} = (9Ae^{\iota k_2 x} + 3Ae^{-\iota k_2 x})/8$$

$$\psi_{III} = 3Ae^{\iota k_3 x}/2$$

And for $n = -72$,

$$\psi_I = Ae^{\iota k_1 x} - Ae^{-\iota k_1 x}/2$$

$$\psi_{II} = (11Ae^{\iota k_2 x} - 7Ae^{-\iota k_2 x})/8$$

$$\psi_{III} = 3Ae^{\iota k_3 x}/2$$

(c) For $n = 8$ it is clearly no phase change and for $n = -72$ there is a phase change of π .

Tutorial 12

Q2

(a) Since we are modelling this as a simple harmonic oscillator we can first find the spring's frequency. We will use $2k$ since the spring is between two masses.

$$f = \frac{1}{2\pi} \sqrt{\frac{2k}{m}} = 1.85 \times 10^{14} \text{ Hz}$$

(b) These states should differ in energy by hf and so this photon has energy 0.76 eV and wavelength 1.62 μm .

Q4

We know that the ground state of the harmonic oscillator is

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

The potential energy is essentially $V(x) = \frac{m\omega^2 x^2}{2}$ and so

$$\langle V \rangle = \int_{-\infty}^{\infty} \frac{m\omega^2}{2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} x^2 \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx$$

Substituting a new variable $t = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$, we can then rewrite this as

$$\langle V \rangle = \frac{\hbar\omega}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t \cdot t e^{-t^2} dt$$

We now apply integration by parts on this and we would get

$$\langle V \rangle = \frac{\hbar\omega}{4\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{\hbar\omega}{4}$$

We would have $\langle E \rangle = \langle KE \rangle + \langle V \rangle$ and since this is the ground state, $\langle E \rangle = E_0 = \frac{\hbar\omega}{2}$ and so $\langle KE \rangle = \frac{\hbar\omega}{4}$. This also shows how the ground state has minimum $\Delta p \Delta x$ (why?) since it is a Gaussian and previously using HUP we had found the ground state energy and there we essentially made $\langle KE \rangle = \langle V \rangle$ for the ground state to minimise energy.

Q5

This is pretty much the same question as Q2 except that we just take m instead of $2m$ since the question states that's the mass of the oscillator.

(a) The wavelength would be $\lambda = 2\pi c \sqrt{\frac{m}{k}} = 1.29 \times 10^{-4} \text{ m}$.

(b) The ground state energy would be $\frac{\hbar c}{2\lambda} = 4.8 \text{ meV}$.

Tutorial 13

Useful formulae for this tut:

$$\text{Maxwell Distribution} \quad f(E) = A \exp\left(-\frac{E}{k_B T}\right)$$

$$\text{Fermi-Dirac Distribution} \quad f(E) = \frac{1}{1 + e^{\frac{E-E_f}{k_B T}}}$$

$$\text{Bose-Einstein Distribution:} \quad f(E) = \frac{1}{e^{\frac{E}{k_B T}} - 1}$$

Statistical Mechanics

Q1

This is a simple PnC question. The ways of choosing 5 balls from the 59 would be $\binom{59}{5}$ and choosing 1 ball from 35 is $\binom{35}{1}$ and since they are independent this gives us the total as $\binom{59}{5} \times \binom{35}{1}$ and the probability would be $(\binom{59}{5} \times \binom{35}{1})^{-1}$.

Q3

Let us first judge the possible energy distributions for the total of $3E$. We must note that no two electrons can be in the same state since they are fermions.

Case 1: all have E , this would have $\binom{10}{3} = 120$ possible configurations since we need to choose 3 states from the 10 available states of energy E .

Case 2: one has $2E$ energy and one has E energy while the remaining one has zero energy. Here we need to choose 1 from each of the levels and so the number of configurations for this would be $\binom{10}{1} \binom{20}{1} \binom{2}{1} = 400$.

There are no other possible cases and so it is case 1 with 23.1% probability and case 2 with 76.9% probability.

Q5

(a) For this situation we can have three at zero energy and one at $9E$ or three at $3E$ and one at zero. Another case is one each at $0, E, 3E$, and $5E$ so the total cases are $\binom{4}{1} + 4! + \binom{4}{1} = 32$ microstates.

(b) If all particles are bosons then they are indistinguishable so we only have three cases each with only one configuration giving us the answer as 3 microstates.

(c) Fermions would prevent any configuration with a state which would have more than one particle so the only case is where all have unique energies so that gives only 1 microstates.

Q6

Due to spin degeneracy 10 states essentially become 20 spin states and so the ways of putting three electrons in these are $\binom{20}{3}$ whereas the ten chairs in a room would be filled in $\binom{10}{3}$ ways by three people.

Q7

Let's call $E_0 = \frac{\pi^2 \hbar^2}{2mL^2}$. So with varying the n_x, n_y, n_z we can see that the lowest five levels are

$3E_0$ with spin degeneracy of 2

$6E_0$ with spin degeneracy of 6

$9E_0$ with spin degeneracy of 6

$11E_0$ with spin degeneracy of 6

$12E_0$ with spin degeneracy of 2

Now we want a total energy of $18E_0$. The possible cases are

Case 1: $3E_0, 3E_0, 12E_0$ which has 2 configurations.

Case 2: $3E_0, 6E_0, 9E_0$ which has $2 \times 6 \times 6 = 72$ configurations.

Case 3: $6E_0, 6E_0, 6E_0$ which has $\binom{6}{3} = 20$ configurations.

And so they have respective probabilities of 0.02, 0.77 and 0.21 approximately.

Density of States and Fermi Energy

Q1

(a) On considering the spin degeneracy the 0 energy level has 2 states, $2E$ has 8 states and $3E$ has 16 states. With this in mind we can see that the microstates for this configuration are $\binom{2}{1} \binom{8}{3} \binom{16}{2} = 13440$.

(b) With the same logic the microstates for $(2, 0, 4)$ are $\binom{2}{2} \binom{8}{0} \binom{16}{4}$ and so the ratio of the probabilities is the ratio of the number of microstates here which would be 13 : 96.

Q3

(a) We know that for a 2D particle in a box $E = \frac{\pi^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2)$. Let us first move to a new k space where $k_x = n_x \pi/L$ and $k_y = n_y \pi/L$ and we define an equivalent $k = \sqrt{k_x^2 + k_y^2}$. In k space a constant energy is represented by a circle centered at origin and we only stay in one quadrant. To find the number of states inside this circle, we would need to quantify the space one state takes, this would simply be $\frac{\pi^2}{L^2}$. So we would get the number of states in this circle to be $N(k) = \frac{\pi k^2/4}{\frac{\pi^2}{L^2}} = \frac{k^2 L^2}{4\pi}$. Since we know $E = \frac{\hbar^2 k^2}{2m}$ we can rewrite N as $N(E) = \frac{mEL^2}{2\pi\hbar^2}$ but since this is a fermi gas this would be doubled by spin degeneracy so $N(E) = \frac{mEL^2}{\pi\hbar^2}$. We now can write

$$g(E) = \frac{dN}{AdE} = \frac{mL^2}{L^2\pi\hbar^2} = \frac{m}{\pi\hbar^2}$$

The $g(E)$ is the density here which would be divided by the area since that is the space over which we are finding the density.

(b) At $T = 0$ there will be no fermion with energy greater than E_F and $f(E)$ would be 1 throughout and so since the total fermions are N we have

$$N = A \int_0^\infty g(E)f(E)dE = \int_0^{E_f} g(E)dE = A \int_0^{E_f} \frac{m}{\pi\hbar^2} dE = \frac{AE_f m}{\pi\hbar^2}$$

So we get $E_f = \frac{N\pi\hbar^2}{Am}$. Also since both $g(E)$ and $f(E)$ are constant over energy, the average energy would simply be the half of maximum hence average energy is $E_f/2$.

Q7

For the blackbody radiation we can model it as photons bound inside a 3D well. For this we can first derive the expression for density of states in momentum space (which is equivalent to the k space). This can be done similar to how we did it for a 2D well in **Q3**. The result you would get is

$$g(p)dp = \frac{4\pi p^2}{h^3} dp$$

Since photons can take two polarization values, we would need to multiply this by 2. Then we can note that for photons $\nu = pc/h$ and so we can convert this to frequency space as

$$g(\nu)d\nu = \frac{8\pi\nu^2}{c^3}$$

Now for bosons we would apply the BE statistics and so

$$n(\nu)d\nu = g(\nu)f(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} d\nu$$

For getting the expression of energy density we simply would multiply this by $h\nu$ and so we get the same expression as **1**

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(\frac{h\nu}{k_B T}) - 1} d\nu$$

Q9

(a) Using the formula for the FD statistics we can see that this would be 0.48, 0.27, 0.12, 4.5×10^{-5} respectively.

(b) When $E = E_f + \delta$, the probability would be $\frac{1}{e^{\delta/k_B T} + 1}$. For this to be 0.25 we need $\delta = \ln(3)k_B T \approx 1.1k_B T$ and for 0.75 probability we would need $\delta = -\ln(3)k_B T \approx -1.1k_B T$

(c) The energy of occupation of state with energy higher than fermi energy by ΔE would have occupancy probability of

$$f(E_f + \Delta E) = \frac{1}{1 + e^{\Delta E/k_B T}} = \frac{e^{-\Delta E/k_B T}}{1 + e^{-\Delta E/k_B T}} = 1 - \frac{1}{1 + e^{-\Delta E/k_B T}} = 1 - f(E_f - \Delta E)$$

And so we have proven the statement.

Q11

Since we have to assume that $f(E)$ is pretty much the same as that of $T = 0$ then probability of occupancy at any state above E_f energy is zero. We can model the electrons to be trapped in a 3d box potential and so $g(E) = C\sqrt{E}$ where C is a constant. The fraction we need is the following

$$\alpha = \frac{\int_{E_f - k_B T}^{E_f + k_B T} g(E)f(E)dE}{\int_0^\infty g(E)f(E)dE} = \frac{\int_{E_f - k_B T}^{E_f} \sqrt{E}dE}{\int_0^{E_f} \sqrt{E}dE} = \frac{E_f^{1.5} - (E_f - k_B T)^{1.5}}{E_f^{1.5}} \approx 1.5 \frac{k_B T}{E_f}$$

Note that had we assumed the $f(E)$ to change very slightly due to a $T > 0$, the $f(E_f)$ becomes 0.5 instantly (why?) and we would have to count for $E_f + k_B T$ in the integral but since the T is small it would approximately just become $0.5((E_f + k_B T)^{1.5} - (E_f - k_B T)^{1.5})$ assuming T is very small which gives the same result as before. For copper this would take value 5.5×10^{-3} at 300K and 0.025 at 1360K. For contribution to specific heat we will need to make note that only electrons in a certain energy range around fermi energy will contribute to the specific heat and by convention we will pick $\pm k_B T$ here as that range since these are the electrons that can move into a free state when given the energy of $k_B T$. And so the internal energy from this would be

$$U = \alpha N_A k_B T = \frac{1.5 N_A (k_B T)^2}{E_f}$$

And so the specific heat contribution would be $C = \frac{\partial U}{\partial T} = 3R \frac{k_B T}{E_f}$. So this gives a contribution of $0.011R$ at 300K and $0.05R$ at 1360K to the specific heat.

Fin.